Tilings generated by Pisot numbers

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(based on work of S. Akiyama)

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Student conference "Combinatorics on Words"
Hojsova Stráž, May 17–21, 2010

Outline

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(β)-expansions

Positional numeration system with base $\beta > 1$.

Greedy β-expansion of $x \in \mathbb{R}_{\geq 0}$ for some $k \in \mathbb{N}$

$$x = x_k\beta^k + x_{k-1}\beta^{x-1} + \cdots + x_1\beta + x_0 + x_{-1}\beta^{-1} + \cdots$$

with $x_i \in \mathcal{A}_\beta = \{0, 1, \ldots, \lfloor \beta \rfloor\}$ and 'greedy condition'

$$\left| x - \sum_{l}^{k} x_i \beta^i \right| < \beta^l \qquad \text{for all } k \geq l.$$

Notation

$$(x)_\beta = \underbrace{x_k\, x_{k-1} \cdots x_1\, x_0}_{\text{integer part}} \bullet \underbrace{x_{-1}\, x_{-2}\, x_{-3} \cdots}_{\text{fractional part}}$$

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Rényi expansion of unity

Beta-transformation

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor, \qquad x \in [0,1)$$

Rényi expansion of 1

$$\mathrm{d}_\beta(1) := t_1 t_2 t_2 \cdots, \qquad t_k = \lfloor \beta T_\beta^{k-1}(1) \rfloor.$$

Proposition (Parry condition)

A word $u \in \mathcal{A}_\beta^*$ is the β-expansion of some $x \in \mathbb{R}_{\geq 0}$ iff $u' <_{\text{lex}} \mathrm{d}_\beta(1)$ for all u' suffixes of u.

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Property (F)

Set of finite β-expansions

$$\mathrm{Fin}(\beta) := \{x \in \mathbb{R}_{\geq 0} \mid (x)_\beta = x_k x_{k-1} \cdots x_1 x_0 \bullet x_{-1} x_{-2} \cdots x_{-l} 0^\omega\}$$

Assume β algebraic integer

$$\mathrm{Fin}(\beta) \subset \mathbb{Z}[1/\beta]_{\geq 0}$$

β has Property (F) if

$$\mathrm{Fin}(\beta) = \mathbb{Z}[1/\beta]_{\geq 0}$$

i.e., $\mathrm{Fin}(\beta)$ closed under arithmetic operations

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Outline

1 β-numeration

2 Tiling construction

3 Tiling properties

4 Examples

Basic setting

$\beta > 1$ a Pisot number of degree $d = r + 2s$ with min. polynomial

$$x^d = a_{d-1}x^{d-1} + a_{d-2}x^{d-2} + \cdots + a_1 x + a_0 \qquad a_i \in \mathbb{Z}$$

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Recall

Pisot number: an algebraic integer $\alpha > 1$ such that all the other roots of its minimal polynomial (called its conjugates) are in modulus less than one.

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Notation: $\beta = \beta^{(1)}$,

$\beta^{(2)}, \ldots, \beta^{(r)}$ real conjugates of β

$\beta^{(r+1)}, \ldots, \beta^{(r+2s)}$ complex conjugates of β such that $\beta^{(r+j)} = \overline{\beta^{(r+s+j)}}$ for $j = 1, \ldots, s$

For $x \in \mathbb{Q}(\beta)$ denote by $x^{(j)}$ its conjugate in $\mathbb{Q}(\beta^{(j)})$

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For $x \in \mathbb{Q}(\beta)$ denote by $x^{(j)}$ its conjugate in $\mathbb{Q}(\beta^{(j)})$, i.e.,

$$x = x_{d-1}\beta^{d-1} + \cdots + x_1\beta + x_0 \mapsto$$

$$\mapsto x^{(j)} = x_{d-1}(\beta^{(j)})^{d-1} + \cdots + x_1\beta^{(j)} + x_0$$

The mapping Φ

Define mapping $\Phi : \mathbb{Q}(\beta) \to \mathbb{R}^{d-1}$

$$\Phi(x) = (x^{(2)}, \ldots, x^{(r)}, \\ \Re(x^{(r+1)}), \Im(x^{(r+1)}), \ldots, \Re(x^{(r+s)}), \Im(x^{(r+s)}))$$

Proposition

Let β be a Pisot number of degree d. Then $\Phi(\mathbb{Z}[\beta]_{\geq 0})$ is dense in $\mathbb{R}^{d-1}$, i.e.,

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Definition of a tile

Notation

- $\mathrm{Fr} \subset \mathcal{A}_\beta^{\mathbb{N}}$ (countable) set of all fractional parts of $x \in \mathbb{Z}[\beta]_{\geq 0}$
- $S_w := \{x \in \mathbb{Z}[\beta]_{\geq 0} \mid \text{fractional part of } (x)_\beta \text{ is } w\}$
- obviously $\mathbb{Z}[\beta]_{\geq 0} = \bigcup_{w \in \mathbf{Fr}} S_w$
- tile $T_w := \overline{\Phi(S_w)}$

Proposition

Let β be a Pisot number. Then the central tile $\mathcal{T} = T_\varepsilon$ is bounded.

Proof. Any $z \in S_\varepsilon$, inspect $\Phi(z) = (\phi_2(z), \dots, \phi_d(z))$

$$z = \sum_{j=0}^{k} z_j (\beta)^j \quad \Rightarrow \quad \phi_i(z) = \sum_{j=0}^{k} z_j (\beta_i)^j \quad \Rightarrow \quad |\phi_i(z)| \text{ bounded}$$

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Properties of tiles, (F) case

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Let β be a Pisot number of degree d with Property (F). Then

$$\mathbb{R}^{d-1} = \bigcup_{w \in \mathrm{Fr}} T_w \,.$$

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Properties of tiles, (F) case

$\mathrm{Inn}(X) =$ the interior of the set X (the union of all open sets in X)

$\partial(X) =$ the set of boundary elements of X

Proposition

Let β be a Pisot unit with (F).
Then for each $x \in S_\varepsilon$ we have $\Phi(x) \in \mathrm{Inn}(T_\varepsilon)$.

Corollary. For each $x \in S_w$ we have $\Phi(x) \in \mathrm{Inn}(T_w)$.
Moreover, $\overline{\mathrm{Inn}(T_w)} = T_w$.

Corollary. $\partial(T_w)$ is closed and nowhere dense in $\mathbb{R}^{d-1}$.

} tiling

Proposition

Let β be a a Pisot unit with (F), $\mathrm{d}_\beta(1) = t_1 \cdots t_{m-1} 1$.
Then each tile T_w is arcwise connected.

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Then for each $x \in S_\varepsilon$ we have $\Phi(x) \in \mathrm{Inn}(T_\varepsilon)$.

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Let β be a a Pisot unit with (F), $d_\beta(1) = t_1 \cdots t_{m-1}1$.
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$\mathrm{Inn}(X)$ = the interior of the set X (the union of all open sets in X)

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Let β be a Pisot number of degree d ~~with Property (F)~~. Then

$$\mathbb{R}^{d-1} = \bigcup_{w \in \mathrm{Fr}} T_w \,.$$

Property (W): For any $x \in \mathbb{Z}[1/\beta]_{\geq 0}$ and $\epsilon > 0$ there exist $y, z \in \mathrm{Fin}(\beta)$ with $|z| < \epsilon$ such that $x = y - z$.

Under the assumption of (W):

- origin is inner point of $\bigcup_{w \in \mathcal{P}} T_w$ for finite $\mathcal{P}$
- $T_w = \overline{\mathrm{Inn}(T_w)}$ for any $w \in \mathrm{Fr}$
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Number of tiles

Proposition

Let β be a Pisot unit such that $\mathrm{d}_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})^\omega$ and m, p are minimal possible. Then there are exactly $m + p$ tiles up to translation.

Proof:

- $\mathscr{D} :=$ set of all suffixes of $\mathrm{d}_\beta(1)$, $\mathscr{D}$ necessarily finite
- $\mathscr{D} = \{d_1, \ldots, d_\ell\}$ such that $d_i <_{\text{lex}} d_{i+1}$, $\forall i = 1, \ldots, \ell - 1$

Consider S_w for some $w \in \mathrm{Fr}$. If

$$d_i \leq_{\text{lex}} w\,, \quad \text{for some } i,\ d_i = t_{q_i} t_{q_i+1} \cdots,\ q_i \geq 2$$

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Conversely, y such restricted integer part $\Rightarrow y \bullet w$ is β-expansion

Thus

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The assertion is showed similarly to 1). □

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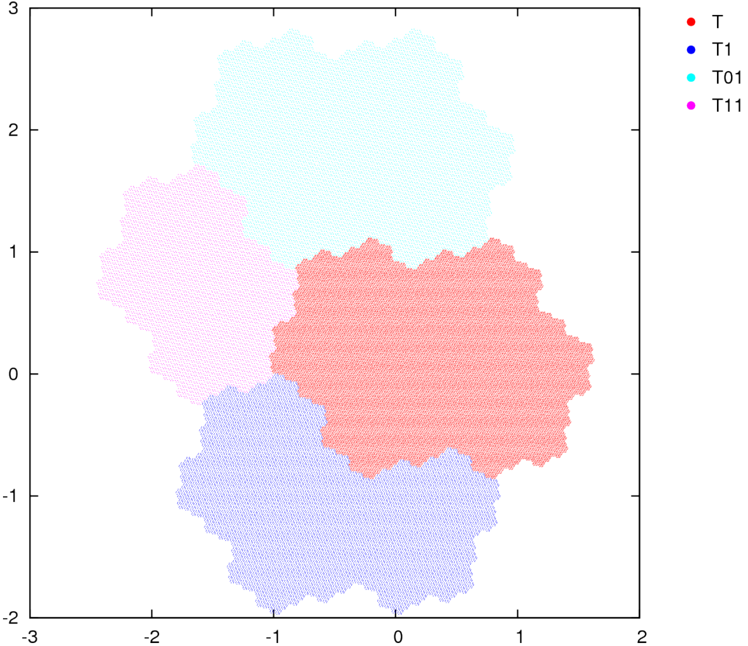
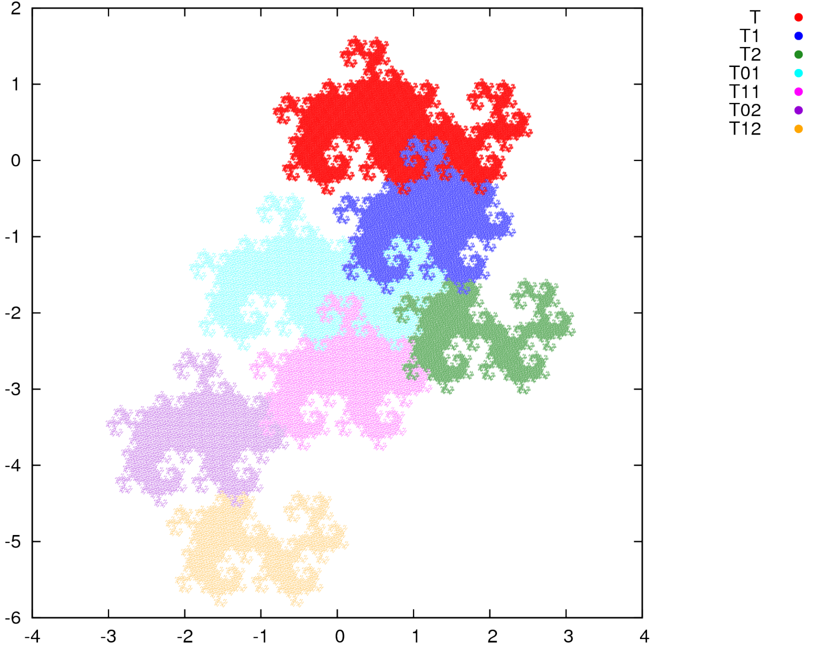
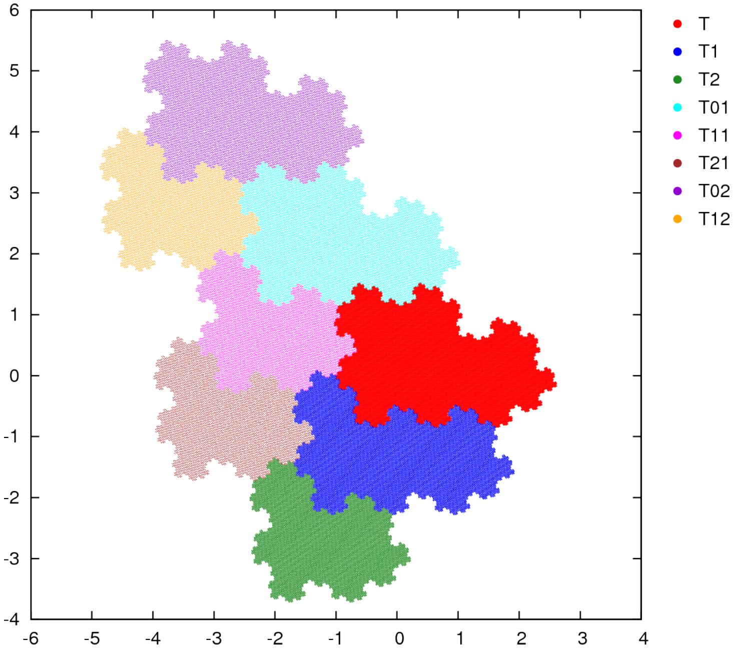
Outline



$$x^3 = x^2 + x + 1$$



$$x^3 = 2x^2 + 1$$



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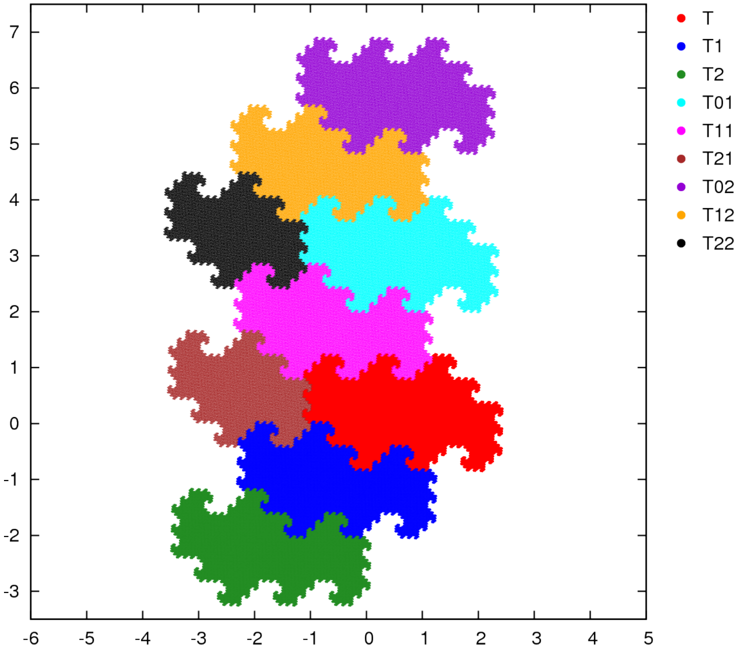

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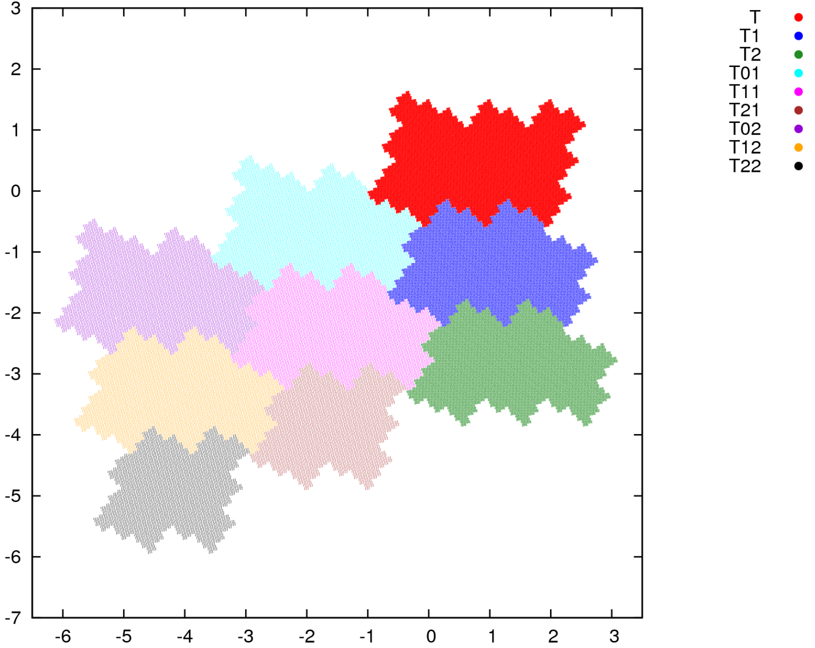
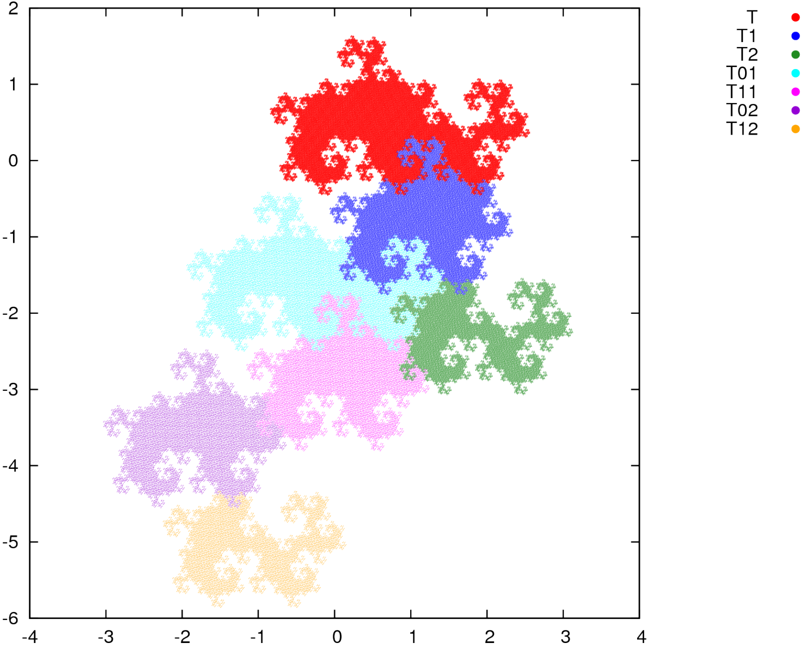


$$x^3 = 3x^2 - x + 1$$



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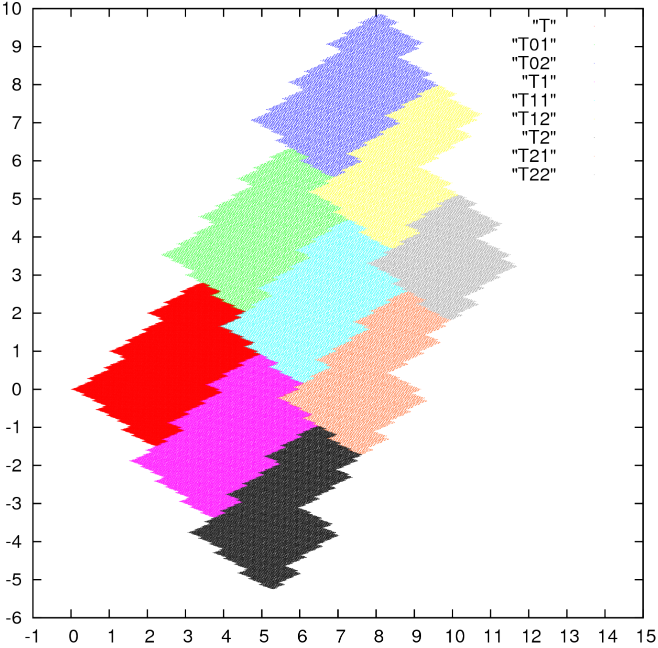

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